- A question on pigeonhole principle (section 6.2).
- Two or three questions on sum rule, product rule, permutation and combination, (sections 6.1 and 6.3).
- At most two questions on solving counting problem using recurrence relations (section 8.1).
- At most three questions on solving recurrence relations (section 8.2).
- One question on inclusion-exclusion principle(section 8.5)
- One question on relations (section 9.1)

How many positive integers between 100 and 999 inclusive are divisible by 3 or 4?

- Divisible by 3: $n = \left\lfloor \frac{999-100+1}{3} \right\rfloor = 300$
- Divisible by 4: $n = \left\lfloor \frac{999-100+1}{4} \right\rfloor = 225$
- Divisible by 3 and 4: $n = \left\lfloor \frac{999-100+1}{3*4} \right\rfloor = 75$

- Divisible by 3 or 4: $300 + 225 - 75 = 450$

How many positive integers between 100 and 999 inclusive are divisible by 3 but not 4?

- Divisible by 3: $n = \left\lfloor \frac{999-100+1}{3} \right\rfloor = 300$
- Divisible by 3 and 4: $n = \left\lfloor \frac{999-100+1}{3*4} \right\rfloor = 75$

- Divisible by 3 but not 4 : $300 - 75 = 225$

How many one-to-one functions are there from a set with five elements to sets with 4 number of elements?

- 0
- A function is one-to-one, if each element only has a **unique** image
- We have 5 elements in the domain and 4 elements in the range (image). Two elements will need to have the same image.

How many one-to-one functions are there from a set with five elements to sets with <u>6</u> number of elements?

- $6 \times 5 \times 4 \times 3 \times 2 = 720$

- The 1st element in the domain have 6 possible ways
- The 2nd element in the domain have 5 possible ways
- The 3rd element in the domain have 4 possible ways
- …

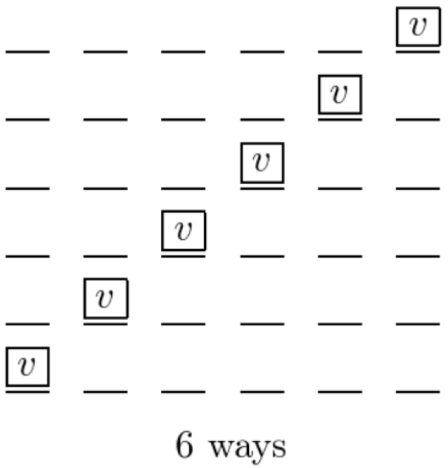
How many numbers must be selected from the set {1, 2, 3, 4, 5, 6, 7, 8} to guarantee that at least one pair of these numbers add up to 9?

- 5
- Because, the are 4 subsets of two elements can add up to 9
  - *{1,8}, {2,7}, {3,6}, {4,5}*

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?

$$n = 6 \times 21^5 \times 5$$



b) exactly two vowels?

$$n = C(6,2) \times 21^4 \times 5^2$$

Hint: The two vowels can be at any locations -> *C(6,2)*

The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

c) at least one vowel?

$$n = 26^6 - 21^6$$

N = (#_of_6_letter_strings) - (#_of_6_letter_strings_with_no_vowel)

d) at least two vowel?

$$n = 26^6 - 21^6 - 6 \times 21^5 \times 5$$

N = (#_of_6_letter_strings) - (#_of_6_letter_strings_with_no_vowel) - (#_of_6_letter_strings_with_1_vowel)

A vending machine dispensing books of stamps accepts only one-dollar coins, $1 bills, and $5 bills.

Find a recurrence relation for the number of ways to deposit *n* dollars in the vending machine, where the order in which the coins and bills are deposited matters.

When n≤5

n=0	a(n)=1 (no deposit)
n=1	a(n)=2
n=2	$a(n)=2^2$
n=3	$a(n)=2^3$
n=4	$a(n)=2^4$
n=5	$a(n)=2^5+1$

When n≥5 (e.g., n=6), we have three choices for next deposit

- Next deposit is **$1 coin**
  - We have a(n-1) way for deposit
- Next deposit is **$1 bill**
  - We have another a(n-1) way for deposit
- Next deposit is **$5 bill**
  - We have a(n-5) way for deposit

Thus, the totally ways is $a_n=2a_{(n-1)}+a_{(n-5)}$ for $n \geq 5$

Solve the recurrence relation together with the initial conditions given.

$a_n = 5a_{n-1} - 6a_{n-2}$ *for* $n \geq 2$, $a_0=1$, $a_1=0$

Step 1: Since it is linear homogeneous recurrence, first find its characteristic equation

Let $a_n=r^2$, $a_{n-1}=r$, $a_{n-2}=1$

Then, $r^2 = 5r-6$

$r^2-5r+6 = 0$

$(r-2)(r-3) = 0$

$r=2$ or $r=3$

Step 2: By **Theorem 1**, the solution of the recurrence relation is $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$

Step 3: Find α_1 and α_2 using initial conditions

$a_0 = \alpha_1 + \alpha_2 = 1$ 1)

$a_1 = 2\alpha_1 + 3\alpha_2 = 0$ 2)

1)*2 $2\alpha_1 + 2\alpha_2 = 2$ 3)

2)-3) $\alpha_2 = -2$

Use α_2 to solve 1), $\alpha_1 = 3$

Step 4: $\boldsymbol{a_n = 3 \bullet 2^n - 2 \bullet 3^n}$

Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$

- Since $a_n = 2a_{n-1} + 3^n$ is a non-homogeneous linear recurrence relation, all solutions are formed as $\{a_n^{(p)} + a_n^{(h)}\}$, $a_n^{(h)}$ is the solutions of the associated homogeneous linear recurrence.
- Step 1, find $a_n^{(h)}$
  - The associated homogeneous linear recurrence is: $a_n = 2a_{n-1}$
  - The characteristic equation: r-2=0 => r=2
  - $a_n^{(h)} = \alpha 2^n$
- Step 2, find $a_n^{(p)}$
  - Consider $a_n^{(p)} = C3^n$
  - Substituting $a_n^{(p)} = C3^n$ in $a_n = 2a_{n-1} + 3^n$
    - $C3^n = 2\,C3^{n-1} + 3^n$
    - $C3^n = 2\,C(3^n/3) + 3^n$
    - $C3^n = 2\,C(3^n/3) + 3^n$
    - $C3^n = (2\,C/3+1)3^n \Rightarrow C3^n = (\frac{2}{3}C+1)3^n \Rightarrow C = \frac{2}{3}C+1 \Rightarrow C=3$
  - Substituting C=3 in $a_n^{(p)} = C3^n$
    - $a_n^{(p)} = 3 \bullet 3^n \Rightarrow a_n^{(p)} = 3^{n+1}$
- Final:
  - $a_n = a_n^{(p)} + a_n^{(h)} = \alpha 2^n + 3^{n+1}$

Given linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-}4 + F(n)$ A) What is the general form of the particular solution guaranteed to exist if **$F(n) = n^3$**

- Step 1, solve the characteristic equation of the associated homogeneous recurrence relation
  - The associated homogeneous recurrence relation, $a_n = 8a_{n-2} - 16a_{n-}4$
  - The characteristic equation of $a_n = 8a_{n-2} - 16a_{n-}4$
    - $r^4 - 8r^2 + 16 = 0$
    - r = 2, -2 or multiplicity 2
- Step 2, find *s* in $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n^1 + b_0)s^n$
  - In our case, t=3, s=1
    - $F(n) = (b_3 n^3)1^n$
- Step 3, apply Theorem 6,
  - s=1, is NOT a root of the associated characteristic equation
  - According to Theorem 6
    - There is a particular solution of the form $(p_3n^3+p_2n^2+p_1n^1+p_0)x1^n =$ $p_3n^3+p_2n^2+p_1n^1+p_0$

Given linear nonhomogeneous recurrence relation
$a_n = 8a_{n-2} - 16a_{n-}4 + F(n)$

- B) Find the *t* and *s* , if $F(n) = n^4 2^n$
  - s=2, t=4
    - $F(n) = (b_4 n^4) 2^n$

- C) Find the *t* and *s* , if $F(n) = n^2 4^n$
  - s=4, t=2
    - $F(n) = (b_2 n^2) 4^n$