Chapter 9.5

Q11: Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

Recall that a relation R on a set A is

(i) Reflexive if $(a,a) \in R$ for all $a \in A$

(ii) Symmetric if $(a,b) \in R$ implies that $(b,a) \in R$

(iii) Transitive if $(a,b),(b,c) \in R$ implies that $(a,c) \in R$

(iv) An equivalence relation if it is reflexive, symmetric and transitive.

Q11 (continue)

Suppose *A* is the set of all bit strings of length three or more. Define a relation *R* on *A* as

$R = \{(x, y) \mid x \text{ and } y \text{ agree in their first three bits}\}$

Clearly $(x, x) \in R$ for all $x \in A$.

Suppose $x, y \in A, (x, y) \in R$

$\Rightarrow x$ and y agree in their first three bits.

$\Rightarrow y$ and x agree in their first three bits.

$\Rightarrow (y, x) \in R$

Q11 (continue)

Suppose $x, y, z \in A, (x, y), (y, z) \in R$

$\Rightarrow$ x and y agree in their first three bits.

And *y* and *z* agree in their first three bits.

$\Rightarrow$ x and z agree in their first three bits.

$\Rightarrow (x, z) \in R$

Hence, *R* is reflexive symmetric and transitive on *A* and therefore, *R* is an equivalence relation on *A*.

Q15: Let R be the relation on the set of ordered pairs of positive integers such that ((a, b), (c, d)) ∈ R if and only if a + d = b + c. Show that R is an equivalence relation.

Suppose A is the set of all ordered pairs of positive integers. Define a relation R on A by

$$R = \{((a,b),(c,d)) \mid a+d = b+c\}$$

Suppose $(a,b) \in A$. Then, because $a+b = b+a$, $((a,b),(a,b)) \in R$ and hence R is reflexive.

Q15 (continue)

Suppose $((a,b),(c,d)) \in R$

$\Rightarrow \; a+d=b+c$

$\Rightarrow \; c+b=d+a$

$\Rightarrow \; ((c,d),(a,b)) \in R$

Therefore, *R* is symmetric

Suppose $((a,b),(c,d)),((c,d),(e,f)) \in R$

$\Rightarrow \; a+d=b+c, c+f=d+e.$ Then,

$a+f=a+d-d+f$

$=b+c-d+f$, because $a+d=b+c$

$=b-d+(c+f)$

$=b-d+d+e,$ because $c+f=d+e$

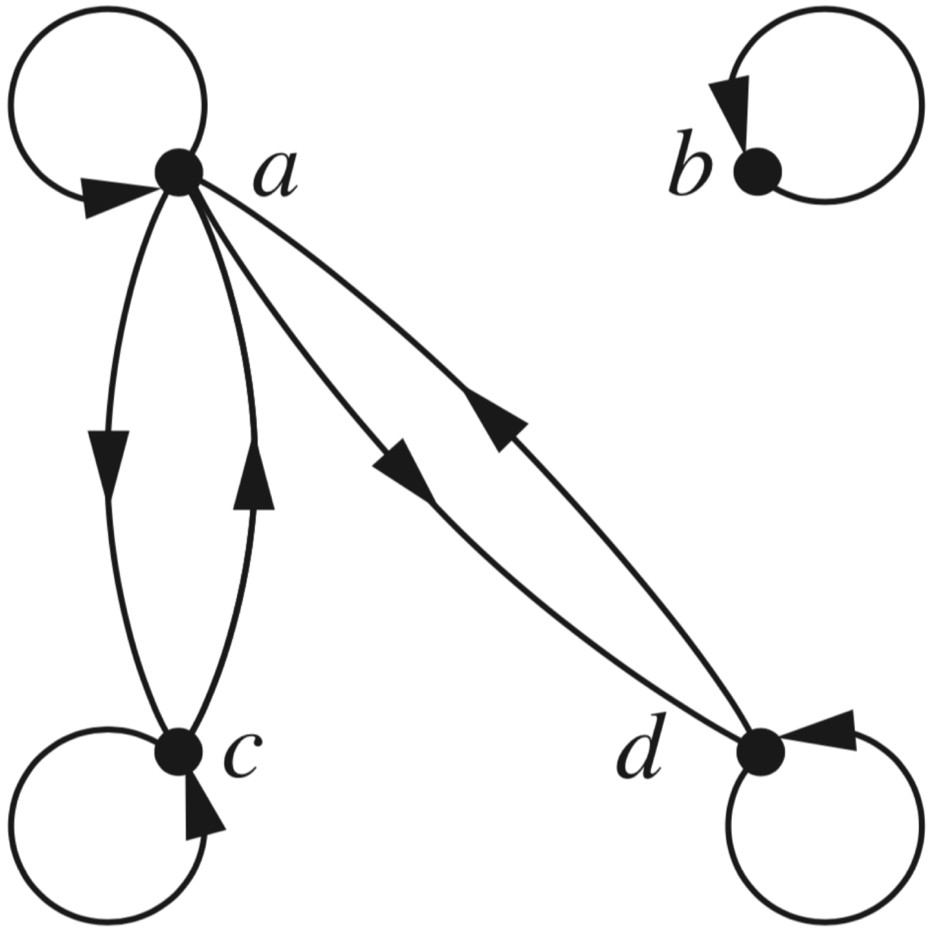
$=b+e$

$\Rightarrow ((a,b),(e,f)) \in R$

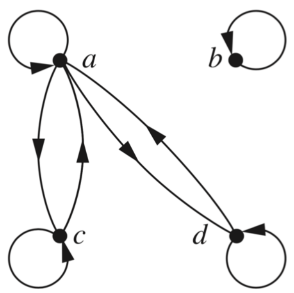
Therefore, *R* is transitive.

Hence, *R* is an equivalence relation on *A*.

Q21: determine whether the relation with the directed graph shown is an equivalence relation.



Q21: determine whether the relation with the directed graph shown is an equivalence relation.

Let G = 

Be the directed graph representing a relation R on {a,b,c,d}.

Then, R is reflexive, symmetric.
But not transitive because there are edges from *c* to *a* and *a* to *d* but no edge from *c* to *d*.

Hence, *R* is not an equivalence relation.

Q25: Show that the relation R on the set of all bit strings such that s R t if and only if s and t contain the same number of 1s is an equivalence relation.

Suppose *A* is the set of all bit strings and *R* is a relation on *A* defined by

$$R\{(x,y) \mid x \text{ and } y \text{ contain the same number of 1s}\}$$

For any $x \in A$, clearly, *x* and *x* contain the same number of 1s.

Therefore, $(x,x) \in R$ for all $x \in A$ and hence R is reflexive.

Q25 (continue)

Suppose $x, y \in A, (x, y) \in R$

$\Rightarrow$ x and y contain the same number of 1s

$\Rightarrow$ y and x contain the same number of 1s

$\Rightarrow$ $(y, x) \in R$

Therefore, R is symmetric.

Suppose $x, y, z \in A, (x, y), (y, x) \in R$

$\Rightarrow$ x and y contain the same number of 1s

y and z contain the same number of 1s

$\Rightarrow$ x and z contain the same number of 1s

$\Rightarrow$ $(x, z) \in R$

Therefore, R is transitive.

Hence, R is an equivalence relation on A.

Q35: What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2 b) 3 c) 6 d) -3

(a) The equivalence class of 2 contains all integers a such that $a \equiv 2 \pmod{5}$.

The integers in this class are those that have a remainder of 2 when divided by 5.

Hence the equivalence class of 2 for this relation is

$$[2]_5 = \{2 + 5c \mid c \text{ is an integer}\}$$

$$= \{\ldots\ldots\ldots\ldots\ldots, -8, -3, 2, 7, 12, \ldots\ldots\ldots\}$$

Q35: What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2 b) 3 c) 6 d) -3

(b) The equivalence class of 3 contains all integers a such that $a \equiv 3 \pmod{5}$.

The integers in this class are those that have a remainder of 3

when divided by 5.

Hence the equivalence class of 3 for this relation is

$$[3]_5 = \{3 + 5c \mid c \text{ is an integer}\}$$

$$= \{\ldots\ldots\ldots\ldots, -7, -2, 3, 8, 13, \ldots\ldots\ldots\ldots\}$$

Q35: What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2 b) 3 c) 6 d) -3

(c) The equivalence class of 6 contains all integers a such that $a \equiv 6(\bmod 5)$.

The integers in this class are those that have a remainder of 1 (6 – 5 = 1)

when divided by 5.

Hence the equivalence class of 6 for this relation is

$$[6]_5 = \{6+5c \mid c \text{ is an integer}\}$$
$$= \{1+5+5c \mid \text{ c is an integer}\}$$
$$= \{1+5d \mid d \text{ is an integer}\}$$
$$= \{\ldots\ldots\ldots\ldots\ldots, -9, -4, 1, 6, 11, \ldots\ldots\ldots\}$$

Q35: What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2 b) 3 c) 6 d) -3

(d) The equivalence class of – 3 contains all integers a such that $a \equiv -3 \pmod 5$.

The integers in this class are those that have a remainder of 2 (5 – 3 = 2)

when divided by 5.

Hence the equivalence class of – 3 for this relation is

$$[-3]_5 = \{-3+5c \mid c \text{ is an integer}\}$$
$$= \{2-5+5c \mid c \text{ is an integer}\}$$
$$= \{2+5d \mid d \text{ is an integer}\}$$
$$= \{\ldots\ldots\ldots\ldots, -8, -3, 2, 7, 12, \ldots\ldots\ldots\}$$

Q41: Which of these collections of subsets are partitions of {1,2,3,4,5,6}?

a) {1,2},{2,3,4},{4,5,6}
b) {1}, {2,3,6}, {4}, {5}
c) {2,4,6}, {1,3,5}
d) {1,4,5}, {2,6}

Recall that a partition of set *S* is a collection of disjoint non empty subsets of *S* that have *S* as their union. Let $S = \{1,2,3,4,5,6\}$

(a) $\{\{1,2\},\{2,3,4\},\{4,56\}\}$ is not a partition of *S* because the subsets $\{1,2\},\{2,3,4\}$ are not disjoint.

Q41: Which of these collections of subsets are partitions of {1,2,3,4,5,6}?

a) {1,2},{2,3,4},{4,5,6}
b) {1}, {2,3,6}, {4}, {5}
c) {2,4,6}, {1,3,5}
d) {1,4,5}, {2,6}

(b) $\{\{1\},\{2,3,6\},\{4\},\{5\}\}$ is a partition of S because it is a collection of disjoint non empty subsets of S that have S as their union.

(c) $\{\{2,4,6\},\{1,3,5\}\}$ is a partition of *S* because it is a collection of disjoint non empty subsets of *S* that have *S* as their union.

(d) $\{\{1,4,5\},\{2,6\}\}$ is not a partition of *S* because $\{1,4,5\}\cup\{2,6\}\neq S$

Chapter 9.6

Q3: Is (S,R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) a=b or a is an ancestor of b?

d) a and b have a common friend?

Q3: (continue)

Suppose S is the set of all people in the world.

(a) $R = \{(a,b) \mid a \text{ is taller than } b\}$ is a relation on S.

Then, R is not reflexive on S because any person a can not be taller than a.

Hence, (S, R) is not a poset.

(b) $R = \{(a,b) \mid a \text{ is not taller than } b\}$ is a relation on S.

Then, R is not anti – symmetric on S because there may be two different persons with the same height.

Hence, (S, R) is not a poset

Q3: (continue)

(c) $R = \{(a,b) \mid a = b \text{ or a is an ancestor of } b\}$ is a relation on *S*.

Clearly, $(a,a) \in R$ for all $a \in S$.

Suppose $a,b \in S, a \neq b, (a,b) \in R$. then, *a* is an ancestor of *b* so that *b* can not be an ancestor of *a* and hence $(b,a) \in R.$.

Suppose $(a,b),(b,c) \in R$. then, $(a,c) \in R$.

Hence, *R* is a partial ordering

Therefore, (S,R) is a poset.

Q3: (continue)

(d) $R = \{(a,b) \mid a \text{ and } b \text{ have a common friend}\}$ is a relation on *S*. if *a* and *b* have a common friend, *b* and *c* have a common friend, then *a* and *c* need not have a common friend.

Therefore, *R* is not transitive.

Hence, (*S*, *R*) is not a poset.

Q15: Find two incomparable elements in these posets.

a) (P({0,1,2}),⊆)

b) ({1,2,4,6,8},|)

The elements a and b of a poset $(P, \leq)$ are called comparable if either $a \leq b$ or $b \leq a$

(a) $(P(\{0,1,2\}), \subseteq)$ is a poset.

Since $\{0\} \not\subseteq \{1\}$ and $\{1\} \not\subseteq \{0\}$, $\{0\}$ and $\{1\}$ are two in comparable elements in the poset $(P(\{0,1,2\}), \subseteq)$

(b) $(\{1,2,4,6,8\})$ is a poset

Since $4 \nmid 6$ and $6 \nmid 4$, 4 and 6 are two incomparable elements in the poset $(\{1,2,4,6,8\},1)$

Q23: Draw the Hasse diagram for divisibility on the set

a) $\{1,2,3,4,5,6,7,8\}$

b) $\{1,2,3,5,7,11,13\}$

c) $\{1,2,3,6,12,24,36,48\}$

The objective is to draw the Hasse diagrams representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on the sets defined in the following subparts.

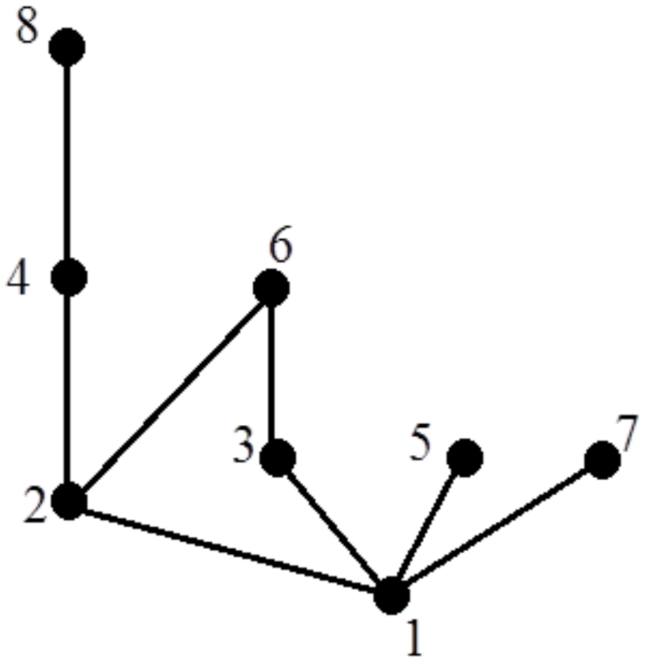
Q23 a)

The objective is to draw the Hasse diagram for the divisibility defined above on the set

$\{1,2,3,4,5,6,7,8\}$.

There exists a path from one number to other when the first one divides the later.

This is shown in the following diagram:



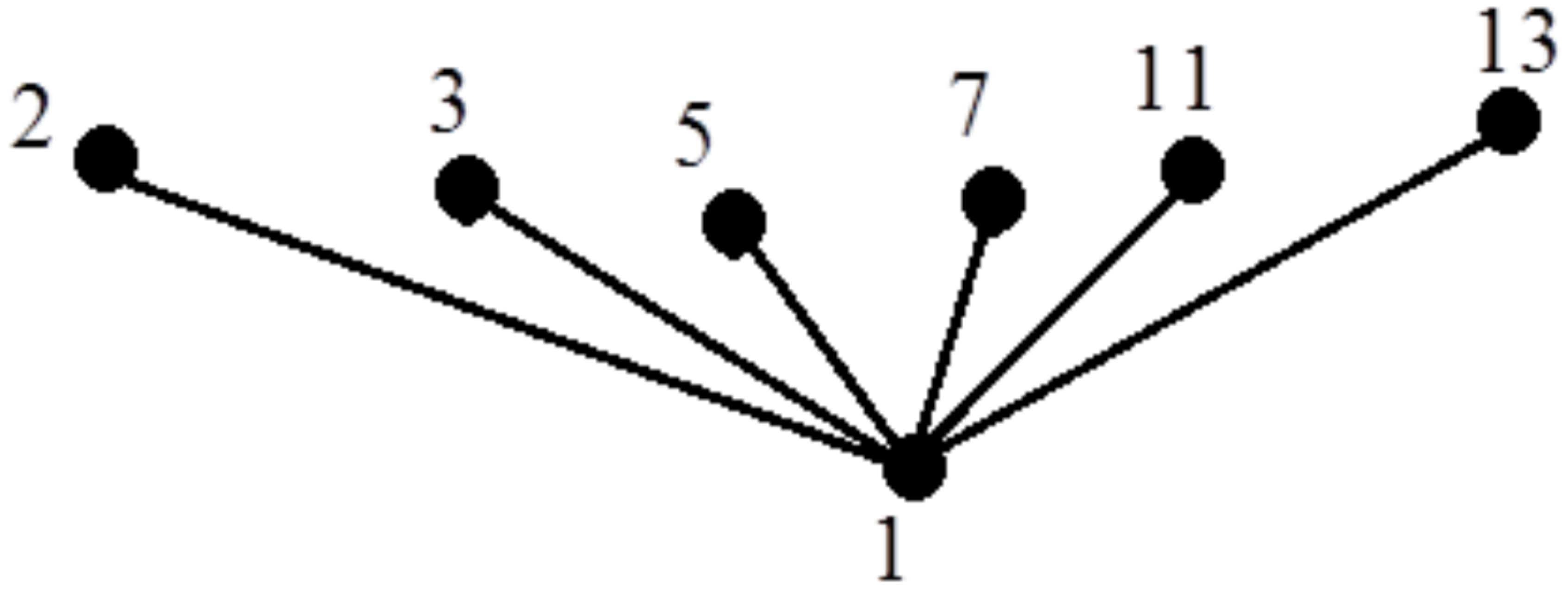
Q23 b)

The objective is to draw the Hasse diagram for the divisibility defined above on the set

$\{1,2,3,5,7,11,13\}$.

There exists a path from one number to other when the first one divides the later.

This is shown in the following diagram since 1 divides $2,3,5,7,11 \text{ and } 13$.



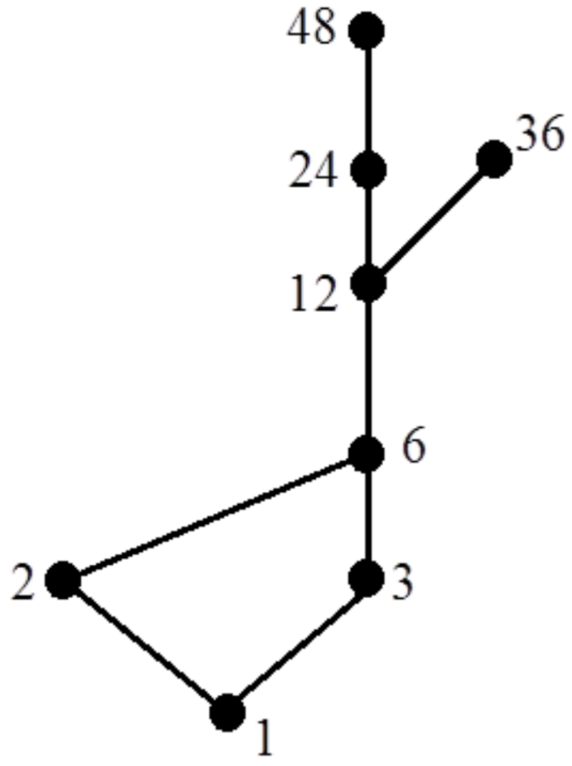
Q23 c)

The objective is to draw the Hasse diagram for the divisibility defined above on the set

$\{1,2,3,6,12,24,36,48\}$.

There exists a path from one number to other when the first one divides the later.

This is shown in the following diagram:



Q33 Answer these questions for the 24, 45}, |).

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of {3, 5}.

f) Find the least upper bound of {3, 5}, if it exists.

g) Find all lower bounds of {15,45}.

h) Find the greatest lower bound of {15,45},if it exists.

Q33 (continue)

Recall the following.

Suppose $(P, \leq)$ is a poset and $A \subseteq P$. then,

(i) An element *a* of *p* is said to be a maximal element if there is no $b \in p$ such that $a < b$.

(ii) An element *a* of *p* is said to be a minimal element if there is no $b \in p$ such that $b < a$.

(iii) An element *a* of *p* is said to be the greatest- element if $b \leq a$ for all $b \in p$.

(iv) An element *a* of *p* is said to be the least – element if $a \leq b$ for all $b \in p$

(v) If *u* is an element of *p* such that $a \leq u$ for all $a \in A$,then *u* is called an upper bound if *A*.

(vi) If l is an element of *p* such that $l \leq a$ for all $a \in A$, then l is called a lower bound of *A*.

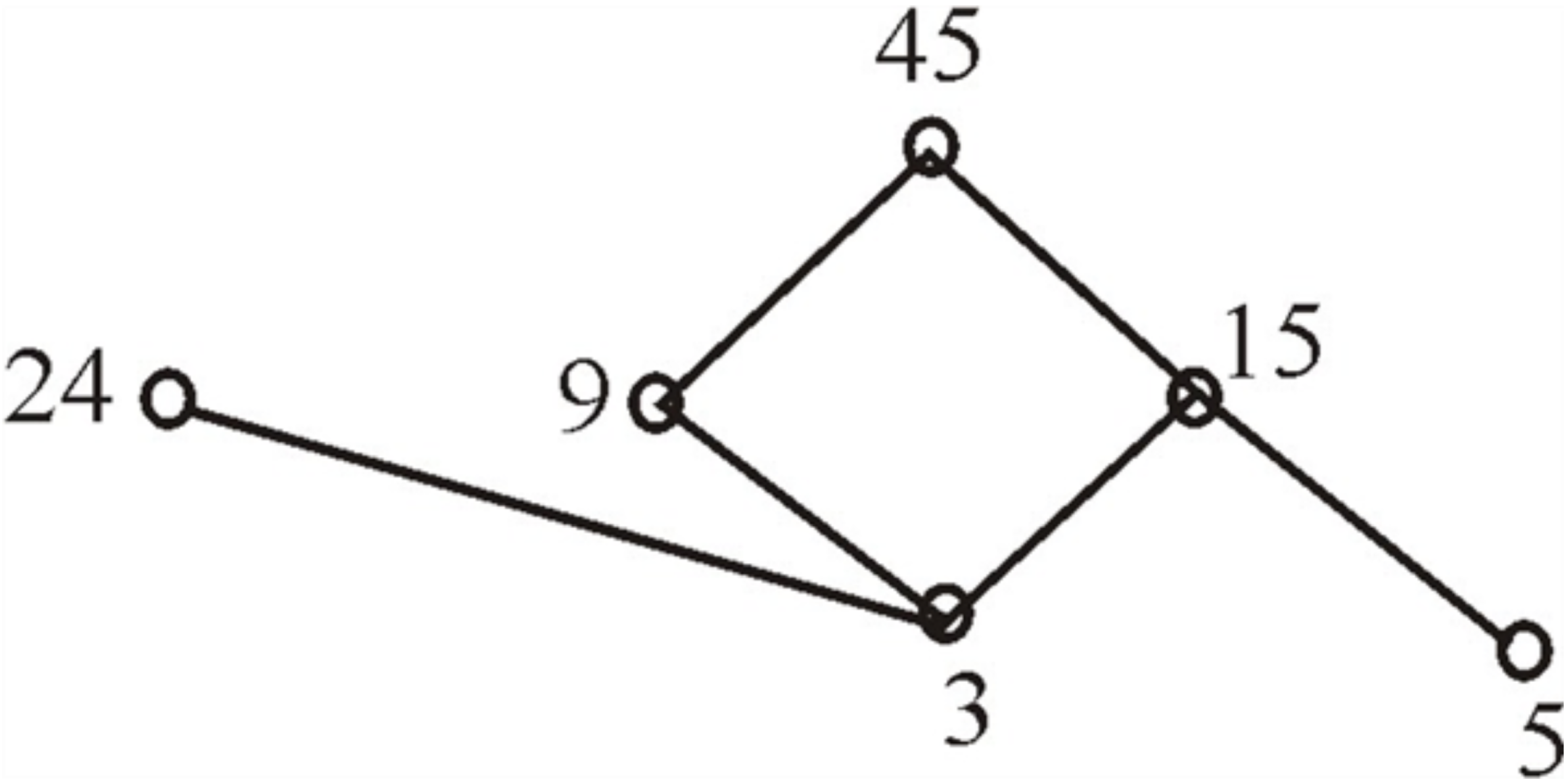
(vii) An element *x* is called the least upper bound of *A* if *x* is an upper bound that is less than any other upper bound of *A*.

(viii) An element *y* is called the greatest lower bound of *A* if *y* is a lower bound that is greater than any other lower bound of *A*.

Q33 (continue)

$(\{3,5,9,15,24,45\}\,|)$ are poset.

The Hasse diagram of the poset is



Q33 (continue)

(a) 24, 45 are maximal elements.

(b) 3, 5 are minimal elements

(c) There is no greatest element.

(d) There is no least element

(e) 15, 45 are upper bounds of $\{3,5\}$

(f) 15 is the least upper bound of $\{3,5\}$

(g) 3,5,15 are lower bounds of $\{15,45\}$

(h) 15 is the greatest lower bound of $\{15,45\}$