CS275 Discrete Mathematics

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Introduction to Proofs
Section 1.7

Direct Proof

- A **direct proof** of a conditional statement $p \rightarrow q$ is constructed where
  - Step 1, assume p is true
  - Step 2, use rules of inference, with the final step showing q must be true

E.g., given a direct proof that if m and n are both perfect squares, then mn is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a=b^2$

- If m and n are perfect squares, there must be integers s and t such that $m=s^2$ and $n=t^2$
- Then, $mn = s^2 t^2 = sstt = (st)(st) = (st)^2$
- Because s and t are two integers, st is an integer
- Thus, mn is also a perfect square

Proof by contraposition

- To prove $p \rightarrow q$ we prove $\neg q \rightarrow \neg p$.
- E.g., prove "if *n* is an integer and 5*n*+2 is odd then *n* is odd"
    - Solution:
        - Since $p \rightarrow q = \neg q \rightarrow \neg p$, we can write the the statement to "if *n* is even, then 5*n*+2 is even"
        - Then $5n+2 = 5(2k)+2 = 10k+2 = 2(5k+1)$
        - Thus, "if *n* is even, then 5*n*+2 is even" is true
        - Because $p \rightarrow q = \neg q \rightarrow \neg p$, we can say if *n* is an integer and 5*n*+2 is odd then *n* is odd" is true

Exercise: prove that if *x* is irrational, then 1/*x* is irrational

- We will use proof by contraposition. The contrapositive is “If 1/x is rational, then x is rational.”
- If 1/x is rational, x is an integer and $x \neq 0$, $1/x \neq 0$
- Since 1/x is rational, 1/x = p/q, p and q are two integers, $q \neq 0$
- Since $1/x \neq 0$, $p/q \neq 0$, thus $p \neq 0$
- 1/x = p/q can be rewritten as x = q/p
- Since x can be written as a quotient of two integers with nonzero denominator, x is rational.
- Thus, “if 1/x is rational, then x is rational.” is true.
- According to $p \rightarrow q = \neg q \rightarrow \neg p$, “if x is irrational, then 1/x is irrational” is true.

Vacuous and Trivial Proofs

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

- For a conditional statement $p \rightarrow q$,
  - if p is false, $p \rightarrow q$ is true
  - if q is true, $p \rightarrow q$ is true
- So, if we can prove "p is false" or "q is true", we can have a quick proof of $p \rightarrow q$ is true.
  - The proof method proves p is false is called ***vacuous*** proofs
  - The proof method proves q is true is called ***trivial*** proofs

Exercise: prove the proposition P(0), where P(n) is the proposition "If *n* is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

- Vacuous proof
- Because
    - p is "if 0 is a positive integer greater than 1"
    - q is "$0^2 > 0$"
    - Since 0 is not a positive integer greater than 1, p is false
    - For a conditional statement $p \rightarrow q$, if p is false, the statement is true.

Exercise: show that these statements about the real number x are equivalent: 1) x is irrational, 2) $3x+2$ is irrational, 3) $x/2$ is irrational

- Statements 1), 2), 3) are equivalent, if 1) and 2) are equivalent, and 1) and 3) are equivalent
- Two statements a and b are equivalent, if $a \rightarrow b$ is true and $b \rightarrow a$ is true

Exercise: show that these statements about the real number *x* are equivalent: 1) *x* is irrational, 2) 3*x*+2 is irrational, 3)*x*/2 is irrational (cont)

- Case 1 1)→2) Proof by contradiction
  - Assume 3x+2 is rational, thus 3x+2 = v/w, v and w are two integers, w≠0
  - 3x+2 = v/w, 3x = (v/w)+2, 3x = (v/w) + 2w/w,

    3x = (v-2)/w, x = (v-2)/3w
  - Since, v and w are two integers and w≠0, v-2 and 3w are two integers. Thus, x is rational. However, x is know as irrational and we have obtained a contradiction. Thus, “3x+2 is rational” is wrong. Thus, “3x+2 is irrational.”

Exercise: show that these statements about the real number *x* are equivalent: 1) *x* is irrational, 2) 3*x*+2 is irrational, 3) *x*/2 is irrational (cont)

- Case 2 2)→1) Proof by contradiction
  - Assume x is rational, then x = v/w, v and w are two integers, w≠0
  - 3x = 3v/w, 3x+2 = (3v/w)+2, 3x+2 = (3v+2w)/w
  - Since, v and w are two integers and w≠0, 3v and 2w are two integers. Thus, 3x+2 is rational. However, 3x+2 is know as irrational and we have obtained a contradiction. Thus, “x is rational” is wrong. Thus, “x is irrational.”

Exercise: show that these statements about the real number *x* are equivalent: 1) *x* is irrational, 2) 3*x*+2 is irrational, 3)*x*/2 is irrational (cont)

- Case 3 1)→3) Proof by contradiction
  - Assume x/2 is rational, thus x/2 = v/w, v and w are two integers, w≠0
  - x/2 = v/w, x = 2v/w
  - Since, v and w are two integers and w≠0, 2v and w are two integers. Thus, x is rational. However, x is know as irrational and we have obtained a contradiction. Thus, “x/2 is rational” is wrong. Thus, “x/2 is irrational.”

Exercise: show that these statements about the real number *x* are equivalent: 1) *x* is irrational, 2) 3*x*+2 is irrational, 3)*x*/2 is irrational (cont)

- Case 4 3)→1) Proof by contradiction
  - Assume x is rational, then x = v/w, v and w are two integers, w≠0
  - x/2 = v/(2w)
  - Since, v and w are two integers and w≠0, v and 2w are two integers. Thus, x/2 is rational. However, x/2is know as irrational and we have obtained a contradiction. Thus, “x is rational” is wrong. Thus, “x is irrational.”

Proof Methods and Strategy
Section 1.8

Exercise: Prove that $n^2+1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

- We can use *proof by exhaustion* or *proof by cases*
  - Find all possible cases, and prove them
- All possible cases:
  - n=1, n=2, n=3, n=4
- Proof:
  - $1^2+1=2 \geq 2^1$
  - $2^2+1=5 \geq 2^2$
  - $3^2+1=10 \geq 2^3$
  - $4^2+1=17 \geq 2^4$

Exercise: Prove or disprove that there is a rational number *x* and an irrational number *y* such that x^y is irrational.

- Let x=2 (rational number) and b=$\sqrt{2}$
- $x^y = 2^{\sqrt{2}}$
    - If $2^{\sqrt{2}}$ is irrational, we have proved the proposition
        - However, it is not straightforward whether $2^{\sqrt{2}}$ is irrational. Let's do more…
- Assume $2^{\sqrt{2}}$ is rational, and let $a = 2^{\sqrt{2}}$, $b=\frac{\sqrt{2}}{4}$
    - $a^b=(2^{\sqrt{2}})^{\frac{\sqrt{2}}{4}} = 2^{\sqrt{2}*\sqrt{2}/4}=2^{2/4} = 2^{\frac{1}{2}} = \sqrt{2}$
- Because $\sqrt{2}$ is irrational, thus "there is a rational number x and an irrational number y such that x^y is irrational" is true
    - *See Example 10 in Sectin 1.7 for proof of $\sqrt{2}$ is irrational*

Exercise: Show that if *n* is an odd integer, then there is a unique integer *k* such that *n* is the sum of *k-2* and *k+3*.

- For this question, we are asked to solve:
  - ***n=(k-2)+(k+3)***
    - *n=2k+1*
    - *n-1=2k*
    - *k*=$\frac{n-1}{2}$
  - Since *n* is odd, *n-1* is even, $\frac{n-1}{2}$ is an integer.
  - Thus, there is a unique integer *k* such that *n* is the sum of *k-2* and *k+3*.

Sets
Section 2.1

Section 2.1, Exercise 5

Determine whether each of these pairs of sets are equal

a) $\{1, 3, 3, 3, 5, 5, 5, 5\}$ $\{5,3,1\}$ True

b) $\{\{1\}\}$ $\{1, \{1\}\}$ False

c) $\emptyset$ $\{\emptyset\}$ False

Section 2.1, Exercise 11

Determine whether each of these statements is true or false

a) $x \in \{x\}$ True

b) $\{x\} \subseteq \{x\}$ True

c) $\{x\} \in \{x\}$ False

d) $\{x\} \in \{\{x\}\}$ True

Section 2.1, Exercise 17
Suppose that A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$. Show $A \subseteq C$

- Suppose that $x \in A$
- Because $A \subseteq B$, $x \in B$
- Because $B \subseteq C$, $x \in C$
- Because $x \in A$ and $x \in C$, $A \subseteq C$

Section 2.1, Exercise 33

Find A^2 if

a) A = {0,1,3} **b) A={1,2,a,b}**

a) {(0,0), (0,1), (0,3), (1,0), (1,1), (1,3), (3,0), (3,1), (3,3)}

b) {(1,1), (1,2), (1,a), (1,b), (2,1), (2,2), (2,a), (2,b), (a,1), (a,2), (a,a), (a,b), (b,1), (b,2), (b,a), (b,b)}

Section 2.1, Exercise 43

Find the truth set of each of these predicates where the domain is the set of integers

a) $P(x): x^2<3$ {-1, 0, 1}

b) $Q(x): x^2>x$ Z-{-1,0,1}

c) $R(x): x>x^2$ ∅