CS275 Discrete Mathematics

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Section 5.2
Strong Induction and Well-Ordering

	Mathematical Induction	Strong Induction
Basis Step	We verify P(1) is true	We verify P(1) is true
Inductive Step	• For all positive integers k, we assume P(k) is true • Show P(k + 1) is true	We show that the conditional statement **$[P(1) \land P(2) \land P(3) \land \cdots \land P(k)] \rightarrow P(k+1)$** is true for all positive integers k.

A modified version of Strong Induction

Let *b* be a fixed integer ant let *j* be a fixed positive integer. To prove P(n) is true for all $n \geq b$, we complete the following two steps:

- Basis step:
    - Verify that the propositions P(b), P(b+1), P(b+2), …, P(b+j) are ture
- Inductive step:
    - Show that the conditional statement $[P(1) \land P(2) \land P(3) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true for every integer **$k \geq b+j$**.

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cents stamps and 5-cent stamps. Proof that $P(n)$ is true for $n \geq 8$.

a) Show that the statements P(8), P(9), and P(10) are true

    a) P(8) is true because 3¢+5¢=8¢

    b) P(9) is true because 3¢+3¢+3¢=9¢

    c) P(10) is true because 5¢+5¢=10¢

Let *P(n)* be the statement that a postage of *n* cents can be formed using just 3-cents stamps and 5-cent stamps. Proof that *P(n)* is true for $n \geq 8$.

b) What is the inductive hypothesis of the proof?

- The statement that using just 3¢ and 5¢ stamps we can form *j*¢ postage for all *j* with $8 \leq j \leq k$

c) What do you need to prove in the inductive step?

- Assume the inductive hypothesis, we can form (k+1)¢ postage using just 3¢ and 5¢ stamps.

Let *P(n)* be the statement that a postage of *n* cents can be formed using just 3-cents stamps and 5-cent stamps. Proof that *P(n)* is true for $n \geq 8$.

d) Complete the inductive step for $k \geq 10$

- Because $k \geq 10$, we know that P(k-2) is true
- We can form (k-2)¢ of postage with 3¢ and 5¢ steps
- (k+1)¢ of postage is (k-2)¢ + 3¢
- Since, $[P(8) \land P(9) \land P(10) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true, the statement is true.

a) Determine which amounts of postage can be formed using just 4¢ and 11¢ stamps

b) Prove your answer using ***mathematical induction***. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer using ***strong induction***.

a) Determine which amounts of postage can be formed using just 4¢ and 11¢ stamps.

- 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, and all values greater than or equal to 30.

b) Prove your answer using *mathematical induction*. Be sure to state explicitly your inductive hypothesis in the inductive step

- Let P(n) be the statement that "we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all n≥30.
- **Basic step:**
  - P(30) = 11+11+4+4 True
- **Inductive hypothesis:**
  - Assume that we can form *k*¢ of postage, we ill show how to form (*k+1*) ¢
- **Inductive step:**
  - If the *k*¢, included an 11¢ stamp, then, replace it by three 4¢ stamps. We can form (k+1)¢ of postage.
  - Otherwise, *k*¢ was formed from just 4¢ stamps. Because k≥30, there must be at least eight 4¢ stamps. Replace the eight 4¢ with three 11¢ stamps. We can form (k+1)¢ of postage.

c) Prove your answer using *strong induction*.

- Let P(n) be the statement that ”we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all n≥30.

- **Basic step:**
  - P(30) = 11+11+4+4
  - P(31) = 11+4+4+4+4+4
  - P(32) = 4+4+4+4+4+4+4+4
  - P(33) = 11+11+11

- **Inductive hypothesis:**
  - P(j) is true for all *j* with 30≤j≤k, where k is an arbitrary integer greater than or equal to 33

c) Prove your answer using *strong induction*.

- Let P(n) be the statement that ”we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all n≥30.

- **Inductive step:**
  - Because k-3≥30, we know that P(k-3) is true.
    - P(j) is true for all j with 30≤j≤k, where k is an arbitrary integer greater than or equal to 33
  - We can form (k-3)¢ of postage with only 4¢ and 11¢ stamps
  - Put one more 4¢ stamp, we can form (k+1)¢ of postage

Which amounts of money can be formed using just \$2 and \$5 bills? Prove your answer using strong induction.

- We can form any amounts expect \$1 and \$3

- Prove
  - Let P(n) be the statement that we can form \$*n* using just \$2 and \$5 bills. We want to prove that P(n) is true for all $n \geq 5$
    - It is clear that we cannot for \$1 and \$3, and it is trivial that we can form \$2 and \$4.

Which amounts of money can be formed using just \$2 and \$5 bills? Prove your answer using strong induction.

- **Base step**
  - P(5) = 5
  - P(6) = 2+2+2
- **Inductive hypothesis**
  - Assume P(j) is true for all *j* with 5≤*j*≤*k*, where *k* is an arbitrary integer grater than or equal to 6. We want to show that P(k+1) is true
- **Inductive step**
  - Because k-1 ≥ 5, we know that P(k-1) is true. We can add a \$2 bill to \$(k-1) to form a \$(k+1) amount.

Section 5.3
Recursive Definitions and Structural Induction

Recursive function

- Basic step:
  - Specify the value of the function at zero
- Recursive step:
  - Given a rule for finding the value of the function at an integer from its value at smaller integers

Find f(1), f(2), f(3), and f(4) if f(n) is defined recursively:

$f(0)=1$

$f(n+1)=3f(n)$

- $f(1) = 3f(0) = 3*1=3$
- $f(2) = 3f(1) = 3*3=9$
- $f(3) = 3f(2) = 3*9=27$
- $f(4) = 3f(3) = 3*27=81$

Find f(1), f(2), f(3), and f(4) if f(n) is defined recursively:

$f(0)=1$

$f(n+1)=f(n)^2+f(n)+1$

- $f(1) = f(0)^2 + f(0) + 1 = 3$
- $f(2) = f(1)^2 + f(1) + 1 = 3^2 + 3 + 1 = 13$
- $f(3) = f(2)^2 + f(2) + 1 = 13^2 + 13 + 1 = 183$
- $f(4) = f(3)^2 + f(3) + 1 = 183^2 + 183 + 1 = 33{,}673$

Find f(2), f(3), f(4), and f(5) if f(n) is defined recursively:

$f(0)=-1$

$f(1)=2$

$f(n+1)=f(n)+3f(n-1)$

- $f(2) = f(1)+3f(0) = 2-3 = -1$
- $f(3) = f(2)+3f(1) = -1+3 * 2= 5$
- $f(4) = f(3)+3f(2) = 5-3*1 = 2$
- $f(5) = f(4)+3f(3) = 2+3 * 5= 17$

Determine whether the proposed definition is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers.

- $f(0)=0, f(n)=2f(n-2)$ for $n \geq 1$ **Not Valid**
- $f(0)=1, f(n)=f(n-1)-1$ for $n \geq 1$ **Valid**

Find the formula of the recursive definition for f(n) when n is a nonnegative integer and prove your formula:

f(0)=1, f(n)=f(n-1)-1 for n≥1

- We list a few cases:
  - f(1) = 0, f(2)=-1, f(3)=-2, f(4)=-3
  - f(n) = 1-n
- Prove:
  - Basic step: f(1) = 1-1 = 0
  - Inductive step:
    - If f(k) = 1-k, then f(k+1) = f(k)-1=1-k-1=1-(k+1)

Find the formula of the recursive definition for f(n) when n is a nonnegative integer and prove your formula:

$f(0)=2$, $f(1)=3$, $f(n)=f(n-1)-1$ for $n \geq 2$

- List a few cases:
  - $f(2) = 2$, $f(3)=1$, $f(4)=0$, $f(5)=-1$
  - $f(n) = 4-n$
- Prove:
  - Basic step: $f(2) = 4-2=2$, $f(3) = 4-3=1$
  - Inductive step:
    - If $f(k) = 4-k$, then $f(k+1) = f(k)-1=(4-k)-1=4-(k+1)$

Prove that **$f1+f3+...+f_{n-1}=f_{2n}$** when *n* is a positive integer

- Let P(n) be "$f1+f3+...+f_{n-1}=f_{2n}$"
- Basic step: P(1) = P(2)
- Inductive step:
  - Assume P(k) is true
  - $f1+f3+...+f_{2k-1}+f_{2k+1} = f_{2k}+f_{2k+1} = f_{2k+2} = f_{2(k+1)}$