CS275 Discrete Mathematics

Gongbo "**Tony**" Liang
Fourth year PhD student in CS
gb.liang@uky.edu
liang@cs.uky.edu

Using truth tables, prove ¬(p∨q)≡¬p∧¬q

p	q	p∨q	¬(p∨q)	¬p	¬q	¬p∧¬q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Using truth tables, prove $p \wedge q \equiv \neg(p \rightarrow \neg q)$

p	q	$p \wedge q$	$\neg q$	$p \rightarrow \neg q$	$\neg(p \rightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	F	T	T	F

Prove that the product of an irrational number and a non-zero rational number is irrational. (Proof by contradiction)

- Assume that *x* is an irrational number and *y* is a non-zero rational number.
- Suppose, *x*•*y* is not irrational
- By the definition of rational, there exists integers *a* and *b* (*b*≠0) such that $y = \frac{a}{b}$
- Since *x* is irrational, *x* cannot be written as the ratio of two integers.
- Since *x*•*y* is not irrational, *x*•*y* is rational. There exists integers *c* and *d* (*d*≠0) such that $xy = \frac{c}{d}$

(next page)

Prove that the product of an irrational number and a non-zero rational number is irrational. (Proof by contradiction)

- Since $y = \frac{a}{b}$, $xy = \frac{c}{d} = x \bullet \frac{a}{b}$
- Multiple each side by $\frac{b}{a}$ (*a*≠0 because *y* is non-zero), $x = \frac{c}{d} \bullet \frac{b}{a}$
- Since *a*, *b*, *c*, *d* are integers, *bc* and *ad* are also integers.
- Thus, $x = \frac{bc}{ad}$, *x* is a rational number, which is in contradiction with the fact that *x* is irrational number.
- Thus, "*x*•*y* is not irrational" is false.
- Thus, *x*•*y* is irrational

Prove that $A \cup (B - A) = A \cup B$

- Let $x \in A \cup (B - A)$
- By definition of union, $x \in A$ or $x \in (B - A)$.
- If $x \in (B - A)$, by set difference $x \in B$ and $x \notin A$.
- But $x \in A$ by previous statement, so $x \in A$ or $x \in B$.
- By definition of union, $x \in (A \cup B)$.

Prove that $A \cap (B - A) = \emptyset$

- Assume $A \cap (B - A) \neq \emptyset$
- There exists an x such that $x \in A \cap (B - A)$
- So, $x \in A$ and $x \in (B - A)$
- If $x \in (B - A)$, $x \in B$ but $x \notin A$
- It is contradiction with each other that $x \in A$ and $x \notin A$
- Thus, $A \cap (B - A) = \emptyset$

Given an example of a function

- f : $Z^+ \longrightarrow Z^+$ that is one-to-one but not on-to.
  - f(x) = 2x+1
    - Not on-to because the image contains only odd integers
- g : $Z^+ \longrightarrow Z^+$ that is on-to but not one-to-one
  - f(x) = $\left\lceil \frac{x}{2} \right\rceil$
- h : $Z^+ \longrightarrow Z^+$ that is one-to-one and on-to. h should not be the trivial function h(x) = x $\forall x \in Z^+$.
  - f(x) = x+1
- f1 : R $\longrightarrow$ R that is strictly increasing
  - f(x) = 5x-3
- f2 : R $\longrightarrow$ R that is increasing but not strictly increasing
  - f(x) = $\left\lfloor \frac{x}{3} \right\rfloor + 1$

Let f:R→R, g:R→R, and h:R→R be defined as follows:

f(x)=x^2−1∀x∈R, g(x)=$\sqrt{x^2+1}$∀x∈R, h(x)=5x−1∀x∈R.

- Define g ∘ f
  - g ∘ f = g(f(x)) = $\sqrt{(x^2-1)^2+1}$
- Define g ∘ f ∘h
  - g ∘ f ∘h = g(f(h(x))) = $\sqrt{((5x-1)^2-1)^2+1}$
- Is it possible to define f^{-1}? If not, why not? If yes, define f^{-1}
  - No, f(x) doesn't have an inverse because it is not one-to-one or onto in R
- Is it possible to define h^{-1}? If not, why not? If yes, define h^{-1}
  - Yes! The inverse of h(x) shows:

y= 5x−1 // replace h(x) with y

5x = y+1 // solve for x

x = (y+1)/5

y = (x+1)/5 // swap x for y

Using mathematical induction prove the following :
For any positive integer *n*,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (n\frac{n+1}{2})^2$$

- Basis step:
  - P(1)
    - Left side: 1^3=1
    - Right side: $(1 \bullet \frac{1+1}{2})^2 = (1 \bullet \frac{1+1}{2})^2 = 1^2 = 1$
- Inductive hypothesis ：
  - $1^3 + 2^3 + 3^3 + \cdots + k^3 = (k\frac{k+1}{2})^2$
- Inductive step:
  - Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = (k\frac{k+1}{2})^2$ is true, we want to show $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = ((k+1)\frac{(k+1)+1}{2})^2$ (see next page)

So,

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (k(k+1)/2)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2$$

$$= (k+1)^2\left[\frac{k^2}{4} + k + 1\right] = \frac{(k+1)^2}{4} \cdot \left[k^2 + 4k + 4\right]$$

$$= \frac{(k+1)^2}{4}(k+2)^2 = \left[(k+1)(k+2)/2\right]^2$$

Determine whether the following definition is valid recursive definition of a function *f* from the set if **non-negative integers** to the set of **integers**. Justify your answer

- $f(0)=1, f(1)=10, f(n)=f(n-1)+f(n-3)$ if $n \geq 2$
  - $f(2) = f(2-1)+f(2-3) = f(1)+f(-1)$
  - Not valid!

Determine whether the following definition is valid recursive definition of a function f from the set if **non-negative integers** to the set of **integers**. Justify your answer

- $f(0)=10, f(1)=-2, f(2)=4, f(n)=f(n-3)+f(n-2)+5$ if $n \geq 3$.
  - $f(3) = f(3-3)+f(3-2)+5 = f(0)+f(1)+5 = 10-2+5 = 13$
  - $f(4) = f(4-3)+f(4-2)+5 = f(1)+f(2)+5 = -2+4 + 5= 7$
  - $f(5) = f(5-3)+f(5-2)+5 = f(2)+f(3)+5 = 4+13+5 = 22$
  - …
  - Valid

Determine whether the following definition is valid recursive definition of a function *f* from the set if **non-negative integers** to the set of **integers**. Justify your answer

- $f(0) = -5$, $f(1) = 5$, $f(n) = n^2 + 10$ if $n \geq 2$.
  - Not valid, because it is not defined recursively
    - Current term is not based on previous term(s)

Define the following function (from the set of **non-negative integers** to the set of **integers**) recursively.

- $f(n)=(n+1)^2\ \forall n\geq 1$
  - f(1) = 4, f(2)=9, f(3) = 16, f(4) = 25, f(5) = 36, f(6) = 49 ...
  - f(1) = 1+3, f(2) = 4+5, f(3) = 9+7, f(4) = 16+9, f(5) = 25+11, f(6) = 36+13 ...

- The recursive function:
  - $f(1) = 4,\ f(n) = f(n-1) + 2n+1\ \forall n>1$

Define the following function (from the set of **non-negative integers** to the set of **integers**) recursively.

- $f(n)=n(n-1)\ \forall\ n \geq 1$
  - f(1) = 0, f(2)=2, f(3) = 6, f(4) = 12, f(5) = 20, f(6) = 30 ...
  - f(2) = 0+2, f(3) = 2+4, f(4) = 6+6, f(5) = 12+8, f(6) = 20+10 ...
  - f(2) = 0+1*2, f(3) = 2+2*2, f(4) = 6+3*2, f(5) = 12+4*2, f(6) = 20+5*2 ...
  - f(2) = f(1) +1*2, f(3) = f(2) +2*2, f(4) = f(3) +3*2, f(5) = f(4) +4*2 ...

- The recursive function:
  - $f(1) = 0,\ f(n) = f(n-1) + 2(n-1)\ \forall n > 1$

Following slides are from previous labs

Given a direct proof that if m and n are both perfect squares, then mn is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a=b^2$

- If m and n are perfect squares, there must be integers s and t such that $m=s^2$ and $n=t^2$
- Then, $mn = s^2 t^2 = sstt = (st)(st) = (st)^2$
- Because s and t are two integers, st is an integer
- Thus, mn is also a perfect square

Prove "if n is an integer and $5n+2$ is odd then n is odd" (use proof by contraposition)

- Solution:
  - Since $p \rightarrow q = \neg q \rightarrow \neg p$, we can write the the statement to "if n is even, then $5n+2$ is even"
  - Then $5n+2 = 5(2k)+2 = 10k+2 = 2(5k+1)$
  - Thus, "if n is even, then $5n+2$ is even" is true
  - Because $p \rightarrow q = \neg q \rightarrow \neg p$, we can say if n is an integer and $5n+2$ is odd then n is odd" is true

Determine whether the following function *f*:ZxZ->Z is onto

(A function f : A → B is called on-to or surjective if for every element b ∈ B, there is some element a ∈ A with f(a) = b
Z = {⋯−2,−1,0,1,2,3,⋯} = the set of all integers, posi- tive and negative)

- f(m,n) = m+n onto
- f(m,n) = m^2 + n^2 Not onto
- f(m,n) = m onto
- f(m,n) = |n| Not onto
- f(m,n) = m-n onto

Cardinality of sets

- Cardinality of finite sets:
    - The number of elements in the set
- Countable sets:
    - A set that either ***finite*** or ***has the same cardinality as the set of positive integers***
    - If S is a infinite set and S is countable
        - $|S|=\aleph_0$ (aleph null)

Given an example of two uncountable sets A and B such that $A \cap B$ is:
a) finite, b) countable infinite, c) uncountable

- a) A = [1,2], B = [3,4]
- b) A = [1,2]∪Z^+, B = [3,4] ∪Z^+
- c) A = [1,3], B = [2,4]

	Mathematical Induction	Strong Induction	Modified Version of Strong Induction
Basis Step	We verify P(1) is true	We verify P(1) is true	We P(b), P(b+1), P(b+2), …, P(b+j) are true
Inductive Step	For all positive integers k, we assume P(k) is true Show P(k + 1) is true	We show that the conditional statement **[P(1)∧P(2)∧P(3)∧⋯∧ P(k)]**→**P(k+1)** is true for all positive integers k.	Show that the conditional statement **[P(1)∧P(2)∧P(3)∧⋯∧P (k)]**→**P(k+1)** is true for every integer **k ≥ b+j.**

a) Determine which amounts of postage can be formed using just 4¢ and 11¢ stamps.

- 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, and all values greater than or equal to 30.

b) Prove your answer using ***mathematical induction***. Be sure to state explicitly your inductive hypothesis in the inductive step

- Let P(n) be the statement that ”we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all n≥30.
- **Basic step:**
  - P(30) = 11+11+4+4 True
- **Inductive hypothesis:**
  - Assume that we can form *k*¢ of postage, we ill show how to form (*k+1*) ¢
- **Inductive step:**
  - If the *k*¢, included an 11¢ stamp, then, replace it by three 4¢ stamps. We can form (k+1)¢ of postage.
  - Otherwise, *k*¢ was formed from just 4¢ stamps. Because k≥30, there must be at least eight 4¢ stamps. Replace the eight 4¢ with three 11¢ stamps. We can form (k+1)¢ of postage.

c) Prove your answer using ***modified version strong induction***.

- Let P(n) be the statement that ”we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all $n \geq 30$.

- **Basic step:**
  - P(30) = 11+11+4+4
  - P(31) = 11+4+4+4+4+4
  - P(32) = 4+4+4+4+4+4+4+4
  - P(33) = 11+11+11

- **Inductive hypothesis:**
  - P(j) is true for all *j* with $30 \leq j \leq k$, where k is an arbitrary integer greater than or equal to 33

c) Prove your answer using ***modified version strong induction***.

- Let P(n) be the statement that ”we can form *n*¢ of postage using just 4¢ and 11¢ stamps. We want to prove that P(n) is true for all n≥30.

- **Inductive step:**
  - Because k-3≥30, we know that P(k-3) is true.
    - P(j) is true for all j with 30≤j≤k, where k is an arbitrary integer greater than or equal to 33
  - We can form (k-3)¢ of postage with only 4¢ and 11¢ stamps
  - Put one more 4¢ stamp, we can form (k+1)¢ of postage