CS275 Discrete Mathematics

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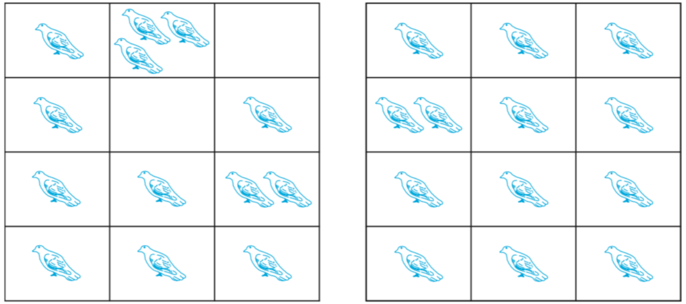
6.2 Pigeonhole Principle

Pigeonhole Principle:

If k is a positive integer and k + 1 or more objects are placed in k boxes, there is at least one box containing two or more objects.



(a) (b)

E.g., we have 13 pigeons and 12 pigeonholes

A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

A) How many socks must he take out to be sure that he has at least two socks of the same color?

3

B) How many socks must he take out to be sure that he has at least two black socks?

14

Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solution:

Because there are four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder.

Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

Solution:

There are four possible subsets of the first eight positive integers can add up to 9: {1,8}, {2,7}, {3,6}, {4,5}

If five integers are selected from the first eight positive integers, by the pigeonhole principle at least two of them come from the same subset.

How many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

4

Because, the are 3 subsets of two elements can add up to 7: *{1,6}*, *{2,5}*, *{3,4}*

Suppose that every student in a discrete math class of 25 students is a freshman, a sophomore, or a junior. Show that there are at **least nine freshmen**, **at least nine sophomores**, or **at least nine juniors** in the class.

Solution:

If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class.

Show that there are **at least six people** in California (population: 37 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.

Solution:

- Totally 6,432,816 “pigeonholes”
  - A person can have 26^3(17,576) possible initials
  - A year can have up to 366 days
- According to Generalized Pigeonhole Principle, there is at least one “pigeonhole” has $\left\lceil \frac{37000000}{6432816} \right\rceil = 6$ people

6.3 Permutation and Combination

Permutation

A permutation of a set of distinct objects is an **ordered** arrangement of these objects.

E.g. Let S = {1,2,3}, the ordered arrangement 3,1,2 is a permutation of S

R-Permutation

An ordered arrangement of r elements of a set is called an *r*-permutation.

E.g. Let S = {1,2,3}, the ordered arrangement 3,2 is a 2-permutation of S

The number of r-permutations of a set with n-elements is denoted by P(n,r).

$$\boldsymbol{P(n,r) = n(n-1)(n-2)\ldots(n-r+1)}$$

If *n* and *r* are positive integers and $(1 \leq r \leq n)$, then

$$\boldsymbol{P(n,r) = \frac{n!}{(n-r)!}}$$

Find the value of each of these quantities.

A) P(8,1)

$8!/7! = 8$

B) P(8,8)

$8! = 40,320$

C) P(10,9)

$10! = 3,628,800$

Combination

An *r*-combination of elements of a set is an **unordered selection** of r elements from the set. The number of r-combinations of a set with n elements is denoted by C(n,r)

The number of r-combinations of a set with n elements where n and r are positive integers such that (0≤r≤n) is given by

$$\boldsymbol{C(n,r) = \frac{n!}{(n-r)!\,r!}}$$

Find the value of each of these quantities.

A) C(8,4)

$$\frac{8!}{(8-4)!\,4!} = \frac{8 * 7 * 6 * 5}{4!} = 70$$

B) C(8,0)

$$\frac{8!}{(8-0)!\,0!} = \frac{8!}{8!} = 1$$

A group contains *n* men and *n* women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution:

$2(n!)^2$

In how many ways can **a <u>set</u> of five letters** be selected from the English alphabet?

Solution:

$$C(26,5) = \frac{26!}{(26-5)!5!} = 65780$$

How many subsets with more than two elements does a set with 100 elements have?

(Recall: if a set *S* has *n* elements, the power set of *S* has 2^n elements)

Solution:

Step 1, we need to find the number of subsets with 2 or less than 2 elements.

$$n' = C(100,2) + C(100,1) + C(100,0) = 4950 + 100 + 1 = 5051$$

Step 2, subtract *n'* from the power set

$$n = 2^{100} - n' = 2^{100} - 5051$$

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

A) are there in total?

$2^{10}=1024$

B) contain exactly two heads?

$$C(10,2)=\frac{10!}{(10-2)!\,2!}=45$$

C) contain at most three tails?

$$C(10,3)+C(10,2)+C(10,1)+C(10,0)=176$$

D) contain the same number of heads and tails

$$C(10,5)=\frac{10!}{(10-5)!\,5!}=252$$

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

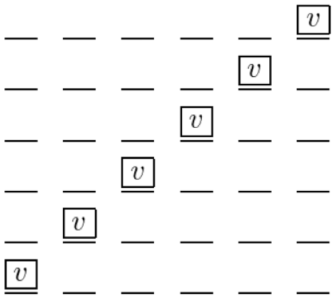
Solution:

$$n = P(8,8)P(9,5)$$

The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?

$$n = 6 \times 21^5 \times 5$$



6 ways

b) exactly two vowels?

$$n = C(6,2) \times 21^4 \times 5^2$$

Hint: The two vowels can be at any locations -> *C(6,2)*

The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

c) at least one vowel?

$$n = 26^6 - 21^6$$

N = (#_of_6_letter_strings) - (#_of_6_letter_strings_with_no_vowel)

d) at least two vowel?

$$n = 26^6 - 21^6 - 6 \times 21^5 \times 5$$

N = (#_of_6_letter_strings) - (#_of_6_letter_strings_with_no_vowel) - (#_of_6_letter_strings_with_1_vowel)