CS275 Discrete Mathematics

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6.4 Binomial Coefficients

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

$$(1 \bullet x + 1 \bullet y)^n = \binom{n}{0} (1 \bullet x)^n + \binom{n}{1} (1 \bullet x)^{n-1} \bullet (1 \bullet y)^1 + \cdots$$

$$(1 \bullet x + 1 \bullet y)^n = \binom{n}{0} (1^n \bullet x^n) + \binom{n}{1} (1^{n-1} \bullet x^{n-1}) \bullet (1^1 \bullet y^1) + \cdots$$

The coefficient of $x^{n-1}y$ in the expansion of $(x+y)^2$ is $\binom{n}{1} 1^{n-1} \bullet 1^1 = \binom{n}{1}$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$(2x - 3y)^{25} = (2x + (-3y))^{25}$

By the binomial theorem, we have $(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$

The coefficient is obtained when j = 13

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}$$

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

$(2x - 3y)^{200} = (2x + (-3y))^{200}$

The coefficient is obtained when j = 99

$$\binom{200}{99} 2^{101}(-3)^{99} = -\frac{200!}{99!\,101!} 2^{101} 3^{99}$$

8.1 Applications of Recurrence Relations

A vending machine dispensing books of stamps accepts only **one-dollar coins**, **$1 bills**, and **$5 bills**.

A) Find a recurrence relation for the number of ways to deposit *n* dollars in the vending machine, where the order in which the coins and bills are deposited matters.

When n≤5

n=0	a(n)=1 (no deposit)
n=1	a(n)=2
n=2	$a(n)=2^2$
n=3	$a(n)=2^3$
n=4	$a(n)=2^4$
n=5	$a(n)=2^5+1$

When n≥5 (e.g., n=6), we have three choices for next deposit

- Next deposit is **$1 coin**
  - We have a(n-1) way for deposit
- Next deposit is **$1 bill**
  - We have another a(n-1) way for deposit
- Next deposit is **$5 bill**
  - We have a(n-5) way for deposit

Thus, the totally ways is $a_n=2a_{(n-1)}+a_{(n-5)}$ for $n\geq5$

A vending machine dispensing books of stamps accepts only **one-dollar coins**, **$1 bills**, and **$5 bills**.

B) What are the initial conditions

n=0	a(n)=1 (no deposit)
n=1	a(n)=2
n=2	$a(n)=2^2$
n=3	$a(n)=2^3$
n=4	$a(n)=2^4$

A vending machine dispensing books of stamps accepts only **one-dollar coins**, **$1 bills**, and **$5 bills**.

C) How many ways are there to deposit $10 for a book of stamps?

$a_n=2a_{(n-1)}+a_{(n-5)}$ for $n\geq5$

n=0	a(n)=1 (no deposit)
n=1	a(n)=2
n=2	$a(n)=2^2=4$
n=3	$a(n)=2^3=8$
n=4	$a(n)=2^4=16$
n=5	$a(n)=2^5+1=33$

n=6	a(n)=2a(5)+a(1)=68
n=7	a(n)=2a(6)+a(2)=136+4=140
n=8	a(n)=2a(7)+a(3)=280+8=288
n=9	a(n)=2a(8)+a(4)=576+16=592
n=10	a(n)=2a(9)+a(5)=1184+33=1217

Find a recurrence relation for the number of bit strings of length *n* that contain a pair of consecutive 0s.

When n<2

n=0	a(n)=0
n=1	a(n)=0

When n≥2 (e.g., n=4), we have three situations

- Bit strings end with 1 (i.e., next bit is 1)
  - We have a(n-1) strings

- Bit strings end with 10
  - We have a(n-2) strings

- Bit strings end with 00
  - We have 2^{n-2} strings

Thus, the totally ways is $a_n = a_{(n-1)} + a_{(n-2)} + 2^{n-2}$ for $n \geq 2$

8.2 Solving Linear Recurrence Relations

Linear recurrences

Each term of a sequence is a linear function of earlier terms in the sequence.

E.g.,

$$a_0 = 1,\ a_1 = 6,\ a_2 = 10$$

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$$

Linear homogeneous recurrences

A linear homogenous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

a_n is expressed in terms of the previous k terms of the sequence, so its degree is k.

Determine which of these are **linear homogeneous** recurrence relations with constant coefficients. Also, find the **degree of those** that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$ **Linear homogeneous! Degree 3**

b) $a_n = 2n \cdot a_{n-1} + a_{n-2}$ **Not linear (n is not a previous term ! Homogeneous (no constant term)!**

c) $a_n = a_{n-1} + a_{n-4}$ **Linear homogeneous! Degree 4**

d) $a_n = a_{n-1} + 2$ **Not homogeneous (2 is a constant term)!**

e) $a_n = (a_{n-1})^2 + a_{n-2}$ **Not liner! Homogeneous!**

f) $a_n = a_{n-2}$ **Linear homogeneous! Degree 2**

Solve the recurrence relation together with the initial conditions given. $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0=1$, $a_1=0$

Step 1: Since it is linear homogeneous recurrence, first find its characteristic equation

Let $a_n=r^2$, $a_{n-1}=r$, $a_{n-2}=1$

Then, $r^2 = 5r-6$

$r^2-5r+6 = 0$

$(r-2)(r-3) = 0$

$r=2$ or $r=3$

Step 2: By **Theorem 1**, the solution of the recurrence relation is $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$

Step 3: Find α_1 and α_2 using initial conditions

$a_0 = \alpha_1 + \alpha_2 = 1$ 1)

$a_1 = 2\alpha_1 + 3\alpha_2 = 0$ 2)

1)*2 $2\alpha_1 + 2\alpha_2 = 2$ 3)

2)-3) $\alpha_2 = -2$

Use α_2 to solve 1), $\alpha_1=3$

Step 4: $\boldsymbol{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

Solve the recurrence relation together with the initial conditions given. $a_n = 4a_{n-1} - 4a_{n-2}$ **for** $n \geq 2$, $a_0 = 6$, $a_1 = 8$

Step 1: Since it is linear homogeneous recurrence, first find its characteristic equation

Let $a_n = r^2$, $a_{n-1} = r$, $a_{n-2} = 1$

Then, $r^2 = 4r-4$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2$

Step 2: By **Theorem 2**, the solution of the recurrence relation is $a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$

Step 3: Find α_1 and α_2 using initial conditions

$a_0 = \alpha_1 = 6$ 1)

$a_1 = 2\alpha_1 + (2 \cdot n \cdot \alpha_2) = 8$ 2)

1)*2 $2\alpha_1 = 12$ 3)

2)-3) $2 \cdot n \cdot \alpha_2 = -4$ $\alpha_2 = -\frac{2}{n}$

Step 4: $a_n = 6 \cdot 2^n + (-\frac{2}{n} \cdot n \cdot 2^n) = 6 \cdot 2^n + (-2 \cdot 2^n) = 6 \cdot 2^n - 2 \cdot 2^n$

Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.

How many different messages can be transmitted in *n* microseconds?

Step 1: Find the recurrence relation

Let a_n be the number of message that can be transmitted in *n* microsecond

The signal requires 1 microsecond for transmittal: a_{n-1}

The signal requires 2 microseconds for transmittal : a_{n-2}

$\mathbf{a_n = a_{n-1} + a_{n-2}}$

Step 2: Find the initial condition

$a_0 = 1$ (the empty message)

$a_1 = 1$ (one message using signal requires 1 microsecond)

Step 3: Solve the recurrence relation together with the initial conditions.

[note: r^2-r-1 = r, r=$\frac{1 \pm \sqrt{5}}{2}$]

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$